Low-Level GPS/DSSS Signal Acquisition Techniques
Tong Detector vs. M-of-N Detector

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INTRODUCTION

The signal acquisition and code-lock detector of a typical DSSS (direct sequence spread spectrum) system tests the magnitude of the in-phase and quadrature phase correlator samples. But if you use the magnitude-squared metric, you must make some small changes to the typical equations in the literature. Here, $Z = I^2+Q^2$ is the statistical measure to determine whether signal is present or not present at the current code phase. All code phases are chipped through until signal is declared.

STATISTICAL MODEL

Most DSSS code-lock detectors use the envelope $Z_{env} = \sqrt{I^2 + Q^2}$ metric, which is outlined in many texts. However, sometimes it is computationally easier to use the squared envelope Z. This changes the corresponding probability of detection and probability of false alarm.

Given that a GPS signal of magnitude A is present with Gaussian noise of variance σ^2, the probability density function (PDF) of Z can be derived as:

$$f_Z(z \mid signal) = \frac{1}{2\sigma^2} \cdot e^{\frac{-1}{2\sigma^2}\left(z+A^2\right)} \cdot I_0\left(\frac{z^{\frac{1}{2}}A}{\sigma^2}\right) \qquad (1.1)$$

where: $I_0(\beta) = \int_0^{2\pi} e^{\beta\cdot\cos(\theta)} d\theta$

is the modified Bessel function of order zero[1]

When no signal is present (A = 0), the PDF of noise only reduces to an exponential PDF:

$$f_z(z \mid nosignal) = \frac{1}{2\sigma^2} \cdot e^{\frac{-z}{2\sigma^2}} \qquad (1.2)$$

For signal acquisition, we compare Z versus a given threshold Vt. If Z is above the threshold, signal is declared present. Otherwise signal is declared absent. This is denoted as a one-shot test.

The one-shot probability of false alarm (Pfa) is the probability that signal is declared when only noise is present. For the $f_Z(z \mid \text{no signal})$ PDF, Pfa is computed by integrating from the threshold Vt to infinity.

$$P_{fa} = \int_{Vt}^{\infty} \frac{1}{2\sigma^2} \cdot e^{\frac{-z}{2\sigma^2}} dz \qquad (1.3)$$

If we desire a certain one-shot Pfa, we can compute the corresponding threshold by evaluating (1.3).

$$P_{fa} = \frac{1}{2\sigma^2} \cdot e^{\frac{-Vt}{2\sigma^2}} \Rightarrow V_t = -2\sigma^2 \ln(P_{fa}) \qquad (1.4)$$

And we can see the threshold Vt is based on the estimated noise variance and the desired Pfa.

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[1] Peebles, P. Probability, Random Variables, and Random Signal Principles, 3rd edition.

Since we derived Vt, we can then compute the one-shot probability of detection (Pd) given the signal strength A and the desired one-shot Pfa by evaluating the integral of $f_Z(z\,|signal)$ from Vt to infinity.

$$P_d = \int_{[-2\sigma^2 \ln(P_{fa})]}^{\infty} \frac{1}{2\sigma^2} \cdot e^{\frac{-1}{2\sigma^2}\left(z+A^2\right)} \cdot I_0\left(\frac{z^{\frac{1}{2}} A}{\sigma^2}\right) dz \quad (1.5)$$

A graphical example of the one-shot probabilities is shown below. With C/No = 35 and a one-shot Pfa = 0.1, we can see the two resulting PDFs and the amount of overlap they have. Evaluating equations (1.4) and (1.5) results in:

For a desired Pfa = 0.1
Vt = $4.6052\sigma^2$
Pd = 0.72333

So, for a given one-shot Pfa and threshold Vt, the one-shot Pd is mainly determined by the signal level "A". As C/No increases, the blue signal+noise PDF moves farther to the right, giving a better one-shot Pd for a given Pfa. As C/No decreases, the blue PDF moves farther to the left, overlapping more with the noise-only PDF and subsequently decreasing the probability of detection Pd. The shaded in areas in the below plot show the area under the PDFs that result in Pfa and Pd.

This is intuitively satisfying, since larger signal levels make it easier to detect signal from noise. Smaller signal levels can be swamped by noise, thereby making it harder to detect the embedded signal.

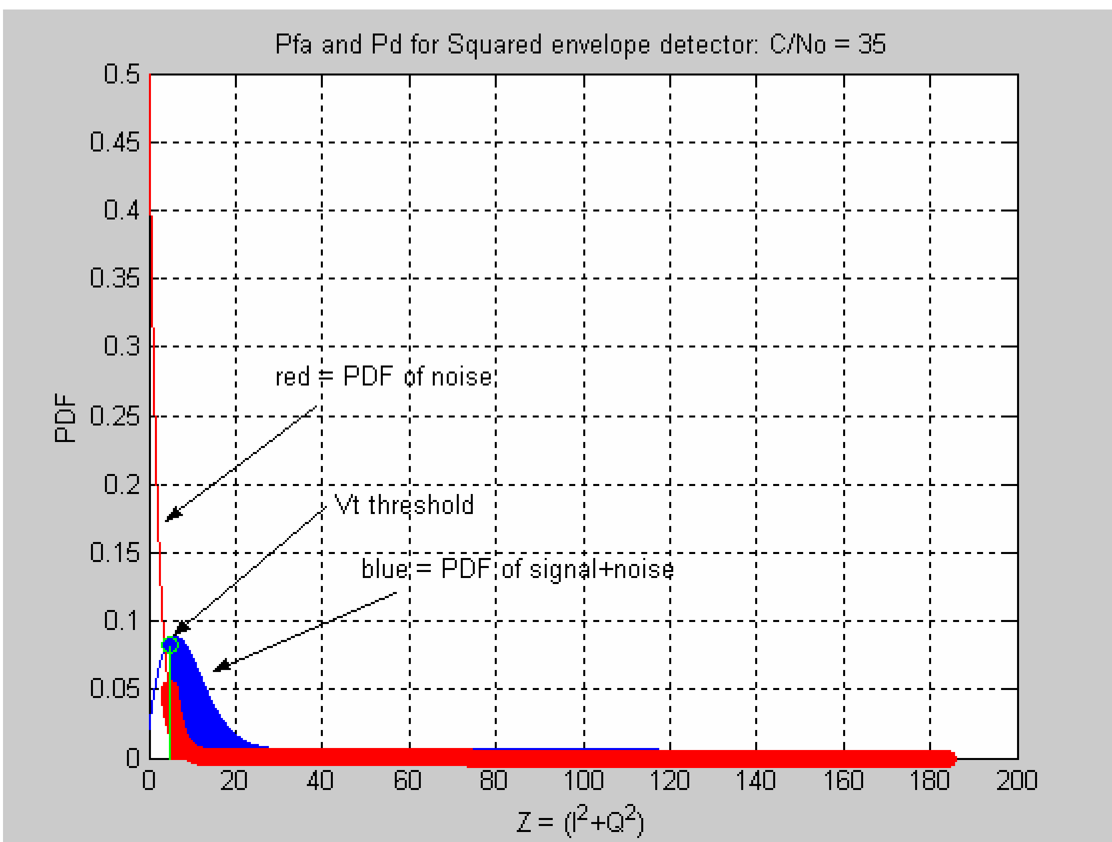

Fig 1: PDF of signal + noise and noise only for one-shot probabilities

SEQUENTIAL DETECTORS FOR SIGNAL ACQUISITION

Most GPS/CDMA receivers do not rely on one-shot detectors since they are not as reliable. The probabilities for signal acquisition are improved by implementing a sequential detector over time. The two detectors most commonly used are the Tong detector and the M-of-N detector.

The satellite's C/No (in dB-Hz) for this analysis is related to the signal amplitude using the equation:

$$\frac{A^2}{2\sigma^2} = 10^{\frac{SNR}{10}} \qquad (1.51)$$

$$SNR = \frac{C}{N_o} + 10\log(T) \qquad (1.52)$$

where: T = pre-detection integration time

$$\therefore A = \sqrt{2\sigma^2 \cdot T \cdot 10^{\frac{C/N_o}{10}}} \qquad (1.53)$$

And we normalize σ^2=1 for the analysis.

We can also derive (1.53) by noting that for a continuous-time integrate and dump operation, the noise-equivalent bandwidth of the integrator can be related to the noise density by:

$$N_o = \sigma^2 T$$

M-of-N Detector:

This type detector is kicked off each time the one-shot Vt threshold has been met. Hence, the overall probability from the "M of N" test is computed from a Bernoulli trial. This results in a discrete binomial distribution.

But, since the first one-shot value had to pass in order to kick off the detector, this first probability is considered independent. Since independence turns conditional probabilities into a multiplication of the probabilities, we get P(D) = P(A)P(B)[1].

Thus, given a one-shot Pfa the overall PFA is a binomial distribution[1] post multiplied by Pfa:

$$P_{FA} = P_{fa} \cdot \sum_{k=M}^{N} C_k^N P_{fa}{}^k (1 - P_{fa})^{N-k} \qquad (1.6)$$

where: $C_k^N = \frac{N!}{k!(N-k!)}$

And the overall PD is:

$$P_D = P_d \cdot \sum_{k=M}^{N} C_k^N P_d{}^k (1 - P_d)^{N-k} \qquad (1.7)$$

Now we can compute and plot the overall PD and PFA curves for varying values of M and N. If we set a desired PFA for the system (which is a design constraint), then we can calculate the theoretical performance of the M-of-N detector over a range of C/No values. This will allow us to determine the probability of signal detection over varying satellite strengths.

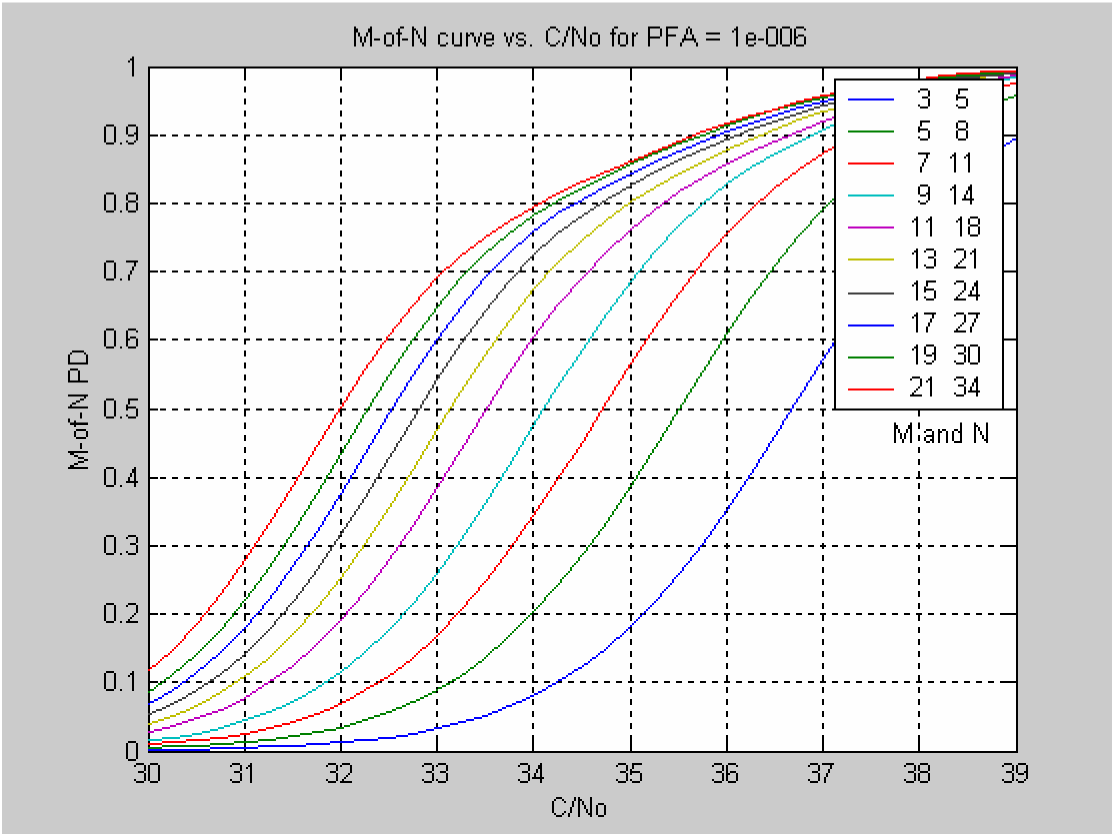

Fig 2: M-of-N detector curves for varying M and N values, PFA = 0.000001

The above plot shows the overall probability of detection vs. C/No using an M-of-N detector. The desired total PFA was set to 1e-6.

Since we declared the total PFA here, the one-shot Pfa and threshold Vt were computed by back substituting the desired PFA value into equation (1.6) and performing a bisection search. This resulted in the one-shot Pfa given M, N, and PFA. Lastly, the one-shot Pd was computed using equation (1.5), and the overall PD probability curve was computed using equation (1.7).

We can see for signals around 32-34 dB-Hz, the fastest "3 of 5" detector has a PD ranging from 1-8%. For these low-signal satellites, this corresponds to very poor acquisition performance over a 3 dB-Hz interval. We can also see for high-strength GPS satellites (40+ dB-Hz), the detector works quite well. This good performance at high C/No is the case with normal GPS acquisition. However, for low C/No SVs the performance of the detector drops off substantially.

Tong Detector:

The Tong detector is a variable-interval sequential detector, whereas the M-of-N detector is a fixed-interval detector. Thus, the M-of-N detector takes longer to run than a corresponding Tong detector.

The Tong detector also exhibits better performance at lower C/No than a similarly configured M-of-N detector. In this discussion, the Tong counter is always initialized to one.

The overall PFA and PD for a Tong detector with Tong threshold N is given by [2]:

[2] Kaplan, E. Understanding GPS Principles and Applications

$$P_{FA} = \frac{\dfrac{(1-P_{fa})}{P_{fa}} - 1}{\left[\dfrac{(1-P_{fa})}{P_{fa}}\right]^{N} - 1} \qquad (1.8)$$

and,

$$P_{D} = \frac{\dfrac{(1-P_{d})}{P_{d}} - 1}{\left[\dfrac{(1-P_{d})}{P_{d}}\right]^{N} - 1} \qquad (1.9)$$

Next is a plot of overall PD for varying C/No values and Tong thresholds. It is used to compare the performance of the Tong detector to the M-of-N detector.

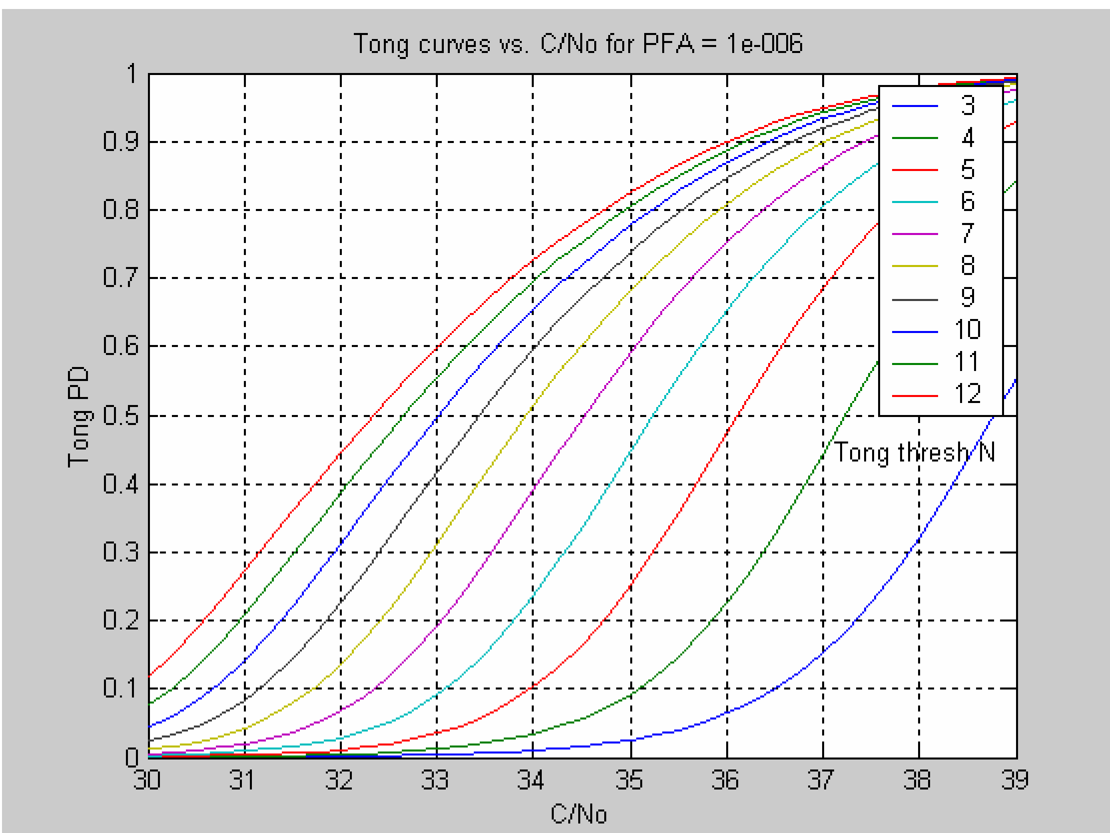

Fig 3: Tong detector PD vs. C/No for varying Tong thresholds, PFA = 0.000001

The desired total PFA was set to 1e-6. Since we constrained the total PFA, the one-shot Pfa and threshold Vt were computed by back substituting the desired PFA value into equations (1.8) and performing a bisection search. This resulted in the one-shot Pfa given the Tong threshold N and PFA. Lastly, the one-shot Pd was computed using equation (1.5), and the overall PD probability curve was computed using equation (1.9).

We can see for lower C/No and large N, the Tong detector performs much better than the M-of-N detector. For example, if we had an SV at 32 dB-Hz, a Tong detector with N = 9 would give us a PD = 22%. We would need to implement a "13 of 21" detector to perform just as well as the Tong detector.

As an additional example, a "3 of 5" detector gives a PD = 2% at a C/No of 32 dB-Hz, where a Tong detector with N = 8 results in a PD = 12%. The improvement at C/No = 33 using a Tong detector with N=8 would result in a 30% increase in detection probability!

As previously stated, for high C/No the Tong detector and M-of-N detector perform similarly. The largest performance gain with the Tong detector is in the C/No range of 32-38 dB-Hz, which is the likely C/No range of a low-level received signal.

APPENDIX

TONG DETECTOR WITH B=2

The standard formula for the PD and PFA for a Tong detector are[2]:

$$P_{FA} = \frac{\dfrac{(1-P_{fa})^B}{P_{fa}{}^B} - 1}{\left[\dfrac{(1-P_{fa})}{P_{fa}}\right]^{N+B-1} - 1}$$

$$P_D = \frac{\dfrac{(1-P_d)^B}{P_d{}^B} - 1}{\left[\dfrac{(1-P_d)}{P_d}\right]^{N+B-1} - 1}$$

Where N is the Tong threshold and B is the Tong initial count.

In the above analysis, it was assumed B=1. However, if you want to decrease PFA while also increasing PD, you can set B=2. The downside is that this decreases the search speed since you start out with a higher initial dismissal count. Therefore, dismissing noise-only code phases takes more time. The upshot is the gain in the overall PD for the lower C/No values.

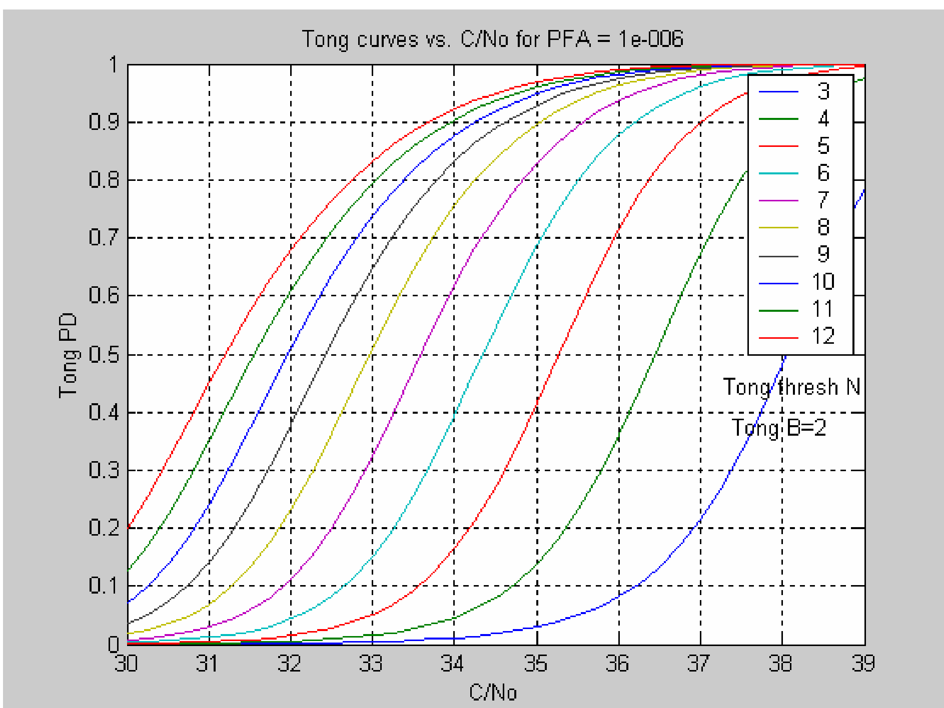

Fig 5: Tong detector curves with B=2

Below is a plot of the PD improvement versus Tong detectors using B=1 and B=2. You can see with an initial Tong count of B=2, we get an absolute PD performance gain around 24% for differing C/No values. This gain is substantial for this analysis since it lies in the low C/No signal strength range, which will increase the PD when attempting to acquire low-level SV signals.

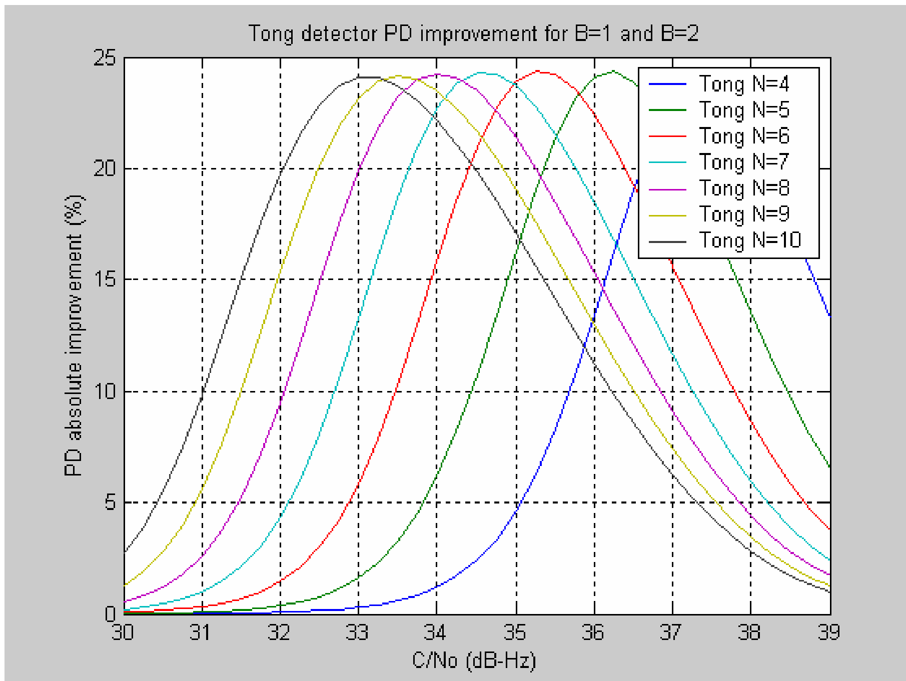

Fig 7: Tong detector improvement for B=2 vs. Tong detector with B=1